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The Petersen Polytopes

David L. Powers
Department of Mathematics and Computer Science
Clarkson University
Potsdam, NY 13676

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1. Background

Let G be a connected graph on n vertices with adjacency matrix $A = \mathbf{adj}(G)$. Godsil (1978) suggested the following construction for a polytope associated with an eigenvalue λ of A. Let $m = \mathrm{mult}(\lambda)$ be the multiplicity of λ and let Z be an n×m matrix of orthonormal eigenvectors associated with λ:

$$Z^T Z = I_m, \quad AZ = \lambda Z.$$

Then the rows of Z, $w_i^T = e_i^T Z$, can be interpreted as points or coordinate vectors of points in m-dimensional euclidean space, R^m. The convex hull of these points, $\mathrm{conv}\{w_1, \ldots, w_n\} = C(\lambda)$, is the polytope of G associated with λ.

If G is vertex-transitive, the polytopes for $\lambda \neq \lambda_1$ have some special properties.

(1) All the w_i have the same euclidean norm $\sqrt{m/n}$ and thus lie on the surface of a sphere in R^m. (Godsil, 1978).

(2) All of the w_i are extreme points of C. If some w_k were not an extreme point of C, it would be a convex combination of other w's. But then w_k would not be an extreme point of the sphere on which they all lie.

(3) The origin is in C if $\lambda \neq \lambda_1$. The eigenvector corresponding to λ_1 is e. Since $\lambda \neq \lambda_1$, $e^T Z = 0$. Hence $\sum_1^n \frac{1}{n} w_i^T = 0$ and 0 is a convex combination of w's.

(4) The polytope has dimension m. Since $0 \in C$, $\dim(\mathrm{aff}\ C) = \dim(\mathrm{span}\ C)$. But the matrix Z has rank m, so $\dim(\mathrm{span}\ C) = m$. (see Brondsted, 1983, p. 10).

(5) The automorphism group of the polytope is transitive on the extreme points. Let i and j be arbitrarily chosen indices. There is

a graph automorphism that takes vertex i to vertex j. There is a corresponding permutation matrix P such that $AP = PA$ and $e_i^T P = e_j^T$. From the former property it follows that $PZ = ZQ$ for some orthogonal matrix Q. (See Godsil, 1978). From the latter it follows that $e_i^T PZ = e_j^T Z = e_i^T ZQ$. Thus Q takes the extreme point i to extreme point j.

Methods for finding facets.

A facet of a convex polytope C is the intersection of C with a supporting hyperplane (Brondsted, p. 53). Since the extreme points (extrema) of a facet F of C are just the extrema of C that lie in F, (p. 45) a facet is completely described by the list of its extrema. In the following we may not distinguish among: (a) a polytope; (b) the set of extrema of the polytope; (c) the set of coordinate vectors of the extrema of the polytope; (d) the set of vertices of the graph G corresponding to the rows of Z that are coordinate vectors of the extrema of the polytope.

Suppose $w_1, \ldots, w_n$ are extrema of an m-polytope $C = \mathrm{conv} \cdot w_1, \ldots, w_n$. Let U be a subset of $1, \ldots, n$ with $U = m$, and let

$$E = \begin{pmatrix} e_{i(1)}^T \\ \vdots \\ e_{i(m)}^T \end{pmatrix}, \quad i(1), \ldots, i(m) \cdot = U$$

1. If EZ is singular, then $W = \mathrm{conv}(w_i : i \cdot U)$ is not a facet.

Proof. If EZ is singular, then $EZx = 0$ for some $x \neq 0$. In order for W to be a facet, we must have all extrema of C on the same side of the hyperplane through W: that is, $Zx \cdot 0$ or $Zx \cdot 0$. But Zx is an eigenvector of A corresponding to an eigenvector other than $_1$

and hence $e^T Z x = 0$. Then we must have $Zx = 0$, but this is impossible since the columns of Z are independent.

2. If EZ is nonsingular, then there is a unique solution of $EZx = e$. Then x defines the hyperplane

$$\{ w : w^T x = 1 \}$$

on which the w_i, $i \in U$, all lie. This hyperplane supports C iff all of C lies on one side of it. Since $0 \in C$, the test becomes this: x defines a support hyperplane iff $Zx \leq e$.

3. If x defines a support hyperplane, the corresponding facet is

$$F = \text{conv} \{ w_i : w_i^T x = 1 \}.$$

This facet includes $\{ w_i : i \in U \}$ by construction.

Note: Suppose x defines a support hyperplane. Then Zx is an eigenvector of A corresponding to λ and having the properties:

(1) $e_i^T Z x = 1$ for at least m values of i. (i.e. Zx has at least m entries equal to 1.)

(2) $e_i^T Z x \leq 1$ for all i.

2. The Petersen Graph

The Petersen graph P is shown in two different views in Figs. 2.1 and 2.2. The first is the "traditional" drawing. The second appeared in Biggs (1973). In referring to vertices of P we use 0 to stand for 10.

The automorphism group of P was first described by Frucht (1936). It is isomorphic to the full symmetric group S_5. Inspection of Fig. 2.1 shows that the vertices of the pentagon are similar (i.e.

Fig. 2.1
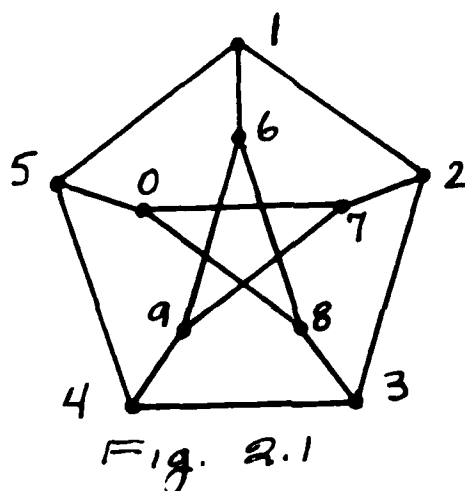
some automorphism takes any one to any other). It is easy to show that an automorphism interchanges the pentagon and the pentagram. Thus P is vertex transitive.

On the other hand, Fig. 2.2 shows that the stabilizer of vertex 1 is transitive on the neighbors of 1, and on the paths of length 2; e.g. some automorphism takes the paht 1,2,3 to the path 1,2,7. Figs. 2.2 and 2.3 reveal that the stabilizer of vertex 1 is transitive on paths of length 3 (but not 4); e.g. the paths 1,2,3,8; 1,2,7,0; 1,2,7,9; 1,2,3,4 can be transformed into each other. Thus P is 3-regular.
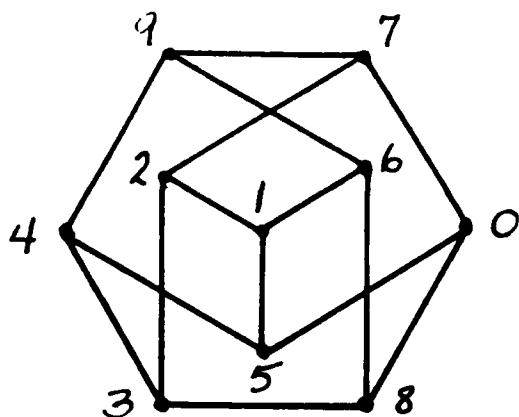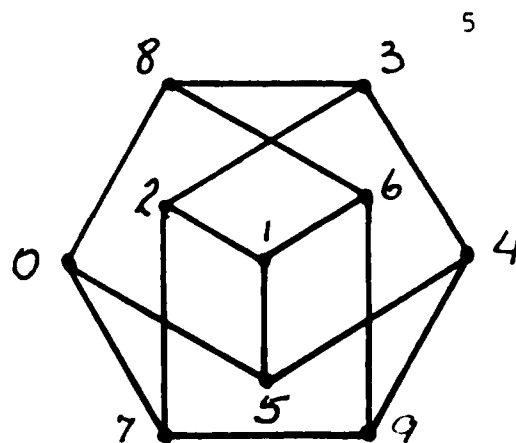
Fig 2.2

Fig. 2.3

We refer to Fig. 2.1 to establish the adjacency matrix of P as

$$A = \begin{pmatrix} C+C^{-1} & I \\ I & C^2+C^{-2} \end{pmatrix}$$

where C is the 5x5 permutation matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which satisfies $C^{-1} = C^T$ and $C^5 = I$.

The minimum polynomial of A can be found as follows. Note that the distance between any two vertices is 1 or 2, and there is a unique path of length d between vertices at distance d. Thus, the off-diagonal entries of A^2 are either 0 or 1, while the diagonal entries are 3. In other words, $A^2 - 3I = adj(\bar{P})$.) Thus

$$A + A^2 = ee^T + 2I. \tag{1}$$

Now $Ae = 3e$, so Eq. (1) multiplied by A yields

$$A^2 + A^3 = 3ee^T + 2A. \tag{2}$$

When ee^T is eliminated between the two equations, one finds

$$A^3 - 2A^2 - 5A + 6I = 0 . \tag{3}$$

Since the polynomial $\lambda^3 - 2\lambda^2 - 5\lambda + 6$ has distinct roots $3,1,-2$, it must be the minimum polynomial of A. If m_1 and m_2 are the multiplicities of eigenvalues 1 and -2 in the characteristic polynomial of A, we have

$$1 + m_1 + m_2 = 10 \tag{4}$$

$$3 + m_1 - 2m_2 = 0, \tag{5}$$

because the characteritic polynomial has degree 10, and $\mathrm{tr}(A) = 0$. Hence $m_1 = 5$, $m_2 = 4$, and the characteristic polynomial is

$$(\lambda-3)(\lambda-1)^5(\lambda+2)^4. \tag{6}$$

Eigenvectors

Several methods are available (and useful) for finding the eigenvectors of the matrix A.

1. Blocks. The matrix

$$A - I = \begin{pmatrix} C+C^{-1}-I & I \\ I & C^2+C^{-2}-I \end{pmatrix} \tag{7}$$

has rank 5, but the first five rows are clearly independent. If a 10x5 matrix of rank 5 can be found that satisfies $[C+C^{-1}-I,I]Z = 0$, then it follows that $[I,C^2+C^{-2}+I]Z = 0$ and hence that Z is a matrix of eigencolumns. Such a matrix is

$$Z = \begin{pmatrix} I \\ I-C-C^{-1} \end{pmatrix}.$$

(8)

Summary: Z above is a 10x5 matrix of independent eigencolumns of A corresponding to the eigenvalue 1.

2. Colorations. Let the vertices of a graph G be partitioned into sets $V_1, V_2, \ldots, V_k$ in such a way that for each pair $i,j \cdot \{1, \ldots, k\}$ each vertex in V_i is adjacent to b_{ij} vertices in V_j. Then the partition is called a coloration of the matrix (Petersdorf & Sachs 1969).

The matrix translation of the condition is this. Let

$$x_{ij} = \begin{cases} 1 \text{ if vertex } i \in V_j \\ 0 \text{ if not.} \end{cases}$$

Then X is the nxk incidence matrix of the partition. The partition is a coloration if and only if

$$AX = XB$$

(9)

for some (integer) matrix B which we call the <u>coloration matrix</u>. In fact $B = [b_{ij}]$ with b_{ij} as above.

The consequences of Eq. (9) are that the characteristics polynomial of B divides that of A, and if $By = \lambda y$, then $A(Xy) = \lambda (Xy)$. If $|V_i| = n_i$, then at least n_i entries of the eigenvector Xy have the same value y_i. The eigenvectors found in this way may lead to the identification of facets of C.

3. The Polytope $C = C(1)$

Theorem 1. If U is the set of vertices lying on a 5-cycle of P then conv$\{w_i: i \in U\}$ is a facet of C.

Proof. Let $V_1 = U$ and $V_2 = V - U$. Then V_1, V_2 form a coloration with coloration matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The eigenvector of B corresponding to the eigenvalue 1 of B is $[1,-1]^T$. Thus one eigenvector z of A corresponding to the eigenvalue 1 has

$$z_i = \begin{cases} 1, & i \in U \\ -1, & i \notin U \end{cases}$$

Note that both U and its complement induce 5-cycles, and that the corresponding facets lie on parallel hyperplanes. #

Theorem 2. If U is the set of vertices lying on a 6-cycle of P, then conv $\{w_i: i \in U\}$ is a facet of C.

Proof. Each 6-cycle is induced by the complement of the closed neighborhood of some vertex. Consider a coloration formed in this way:

$$V_1 = \{k\}, \quad V_2 = \{j: \text{dist}(k,j) = 1\},$$

$$V_3 = \{j: \text{dist}(k,j) = 2\}.$$

The corresponding coloration matrix is

$$B = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

with eigenvalues and eigenvectors

$$\lambda_1 = 3 \qquad\qquad \lambda_2 = 1 \qquad\qquad \lambda_3 = -2$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \qquad\qquad \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix}$$

Thus one eigenvector of A corresponding to the eigenvalue 1 has

$$z_i = \begin{cases} -3 & i \cdot V_1 \\ -1 & i \cdot V_2 \\ 1 & i \cdot V_3 \end{cases}$$

Therefore, the points w_i, $i \in U = V_3$ all lie on a hyperplane of support and thus form a facet. #

<u>Theorem 3</u>. There are no further facets of P.

To prove this theorem, we first need a lemma.

<u>Lemma 1</u>. Let $U = V$, $|U| \geq 5$. Then $\langle U \rangle$ contains a path on three vertices (P_3) or two paths on two vertices $(2P_2)$ or a claw $(K_{1,3})$.

<u>Proof</u>. We may assume the vertex $1 \in U$. First consider four disjoint cases, according to the number of neighbors of 1 in U.

1.1 No neighbor of 1 is in U. Then (see Fig. 2.2) four vertices from $\{3,4,9,7,10,8\}$ are in U and induce a P_3 or a P_4.

1.2 One neighbor of 1 is in U. We may assume $2 \in U$, $5,6 \notin U$. Then decompose further according to the presence in U of the neighbors of 2:

1.2.1 Neither 3 nor 7 in U. Then three of $\{4,9,8,10\}$ are in U and induce a P_2, as do 1 and 2.

1.2.2 One of 3 or 7 in U. Then vertices 1,2 and 3 or 7 induce a P_3.

1.2.3 Both 3 and 7 in U. Then vertex 2 and all its neighbors are in U, inducing a $K_{1,3}$. #

Proof of Theorem 3. Let U be a set of five vertices. We wish to show that the corresponding w's lie on a hyperplane of support iff the vertices lie on a 5-gon or a 6-gon of P. We accomplish this by suitably normalizing U and checking all cases.

We have three (nondisjoint) cases:

(a) $K_{1,3} \subseteq <U>$

(b) $P_3 \subseteq <U>$

(c) $2P_2 \subseteq <U>$

In case (a) we may assume that

$$U = \cdot 1,2,5,6,4 \cdot.$$

The last vertex could be any one at distance 2 from vertex 1; other sets are isomorphic to this one.

In case (b) we may assume that $U = \cdot 1,2,5,x,y \cdot$ and $6 \notin U$. Thus x,y must be drawn from $\{3,4,9,7,10,8\}$. In Table 3.1 we organize the choices by the number of vertices in U that come from the set $\{3,4,7,10\}$ of neighbors of 2 or 5.

Table 3.1

U	# neighbors	Induces
{1,2,5,8,9}	none	$P_3 \cup 2P_1$
{1,2,5,3,8}	one	P_5
{1,2,5,3,9}	one	$P_4 \cup P_1$
{1,2,5,3,4}	two	C_5
{1,2,5,3,7}	two	*
{1,2,5,3,10}	two	P_5

*isomorphic to case a

In case (c), we may assume $1,2 \in U$, $5,6 \notin U$. Now organize by the number of neighbors of 2 in U_1 as shown in Table 3.2.

Table 3.2

U	# neighbors	Induces
$\{1,2,3,7,8\}$	two	*
$\{1,2,3,4,8\}$	one	*
$\{1,2,3,4,9\}$	one	P_5
$\{1,2,3,4,10\}$	one	**
$\{1,2,3,4,7\}$	one	*
$\{1,2,4,9,8\}$	none	$2P_2 \cup P_1$

* isomorphic to case a

**isomorphic to base b3

Note that each of the three U's that induce a P_5 is 5/6 of a 6-cycle and thus each is associated with a facet. Similarly b4 is a 5-cycle. It remains to test the four U's shown in Table 3.3.

Table 3.3

U	Case	Numerical Result
$\{1,2,4,5,6\}$	a	Dependent
$\{1,2,5,8,9\}$	b1	Dependent
$\{1,2,3,5,9\}$	b3	#4 outside
$\{1,2,4,8,9\}$	c6	#3, #6 outside

To confirm the numerical results, use

$$Z = \begin{pmatrix} I \\ I-C-C^{-1} \end{pmatrix}.$$

We solve $EZn = e$ with E corresponding to $\{1,2,3,5,9\}$. The augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{pmatrix}$$

and $n = [1,1,1,3,1]^T$, making $Zn = [1,1,1,3,1,-1,-1,-3,1,-3]^T$.

Clearly vertex 4 is on the "outside." Similarly for $U = \cdot 1,2,4,8,9$

$$[Ez,e] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix}$$

$n = [1,1,3,1,-3]^T$ and

$Zn = [1,1,3,1,-3,3,-3,1,1,-5]^T$. #

To confirm one case of observed dependence we have this theorem.

Theorem 4. Let $AZ = \lambda Z$, where $A = adj(G)$. Then the rows of Z corresponding to any vertex and its neighbors are linearly dependent.

Proof. Consider row i of $AZ = \lambda Z$:

$$e_i^T AZ = \sum_{j@i} e_j^T Z = \lambda e_i^T Z.$$

Thus $\lambda e_i^T Z - \sum_{j@i} e_j^T Z = 0$. #

In the set $U = \{1,2,4,5,6\}$ are the vertex 1 and its three neighbors, so the five corresponding rows of Z have rank 4.

Other cases of dependence can be explained by this theorem.

Theorem 5. Let $AZ = \lambda Z$ where A is the adjacency matrix of Petersen's graph. If U is a set of independent vertices in P, then the corresponding rows of Z are dependent.

Proof. By transitivity, one may take $\cdot 2,5,8,9 \cdot$ as the set U. The corresponding rows of

$$Z = \begin{bmatrix} I \\ I-C-C^{-1} \end{bmatrix} \qquad \text{are} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

and clearly $[1,1,1,1]Z = 0$.   #

The 3-faces

Once the facets of C have been found, the remaining faces can be found by intersecting sets of extrema. Suppose F_1 and F_2 are two different facets. If the intersection of the set of extreme points of F_1 with that of F_2 has at least four elements, then those are extrema of a 3-face. The pattern of intersections is as follows.

(a) Each 6-element facet has extrema corresponding to a 6-gon in P, which is induced by the complement of the closed neighborhood of some vertex. If a 6-element facet is induced by $V - \overline{N}(i)$, let it be called S_i (S for six). Then S_i and S_j have four extrema in common if and only if $i \, \delta \, j$ in P. In this way, fifteen 3-faces are found, each corresponding to a set of four vertices that induce $2P_2$ in P.

(b) If two 5-cycles have exactly two successive edges in common, the remaining edges form a 6-cycle. Each 6-cycle in P is formed in this way by three different pairs of 5-cycles. (See Fig. 2.2). Thus, each 6-element facet of C has four extrema in common with six different 5-element facets. Vice versa, each 5-element facet has four extrema in common with five 6-element facets. In this way, 60 more 3-faces are found, each corresponding to a set of four vertices that induce a P_4 in P.

(c) No 5-element facet intersects another in four extrema. Each 5-element facet, having five extrema in four dimensions, is a simplex and needs no further description. However, each 6-element facet has

nine neighbors: three 6-element, and six 5-element facets. To under-
stand more clearly these facets, consider the polytope S cor-
responding to $U = {1,2,3,8,0,5}$, which is the complement of $\overline{N}(9)$.
Its intersections with four extrema are in Table 3.4.

Table 3.4

Name	Intersection	With facet
80	1,2,3,5	1,2,3,4,5
50	1,2,3,8	1,2,3,6,8
38	1,2,5,0	1,2,5,7,0
23	1,5,8,0	1,5,6,8,0
1:	2,3,8,0	2,3,7,8,0
12	3,5,8,0	3,4,5,8,0
35	1,2,8,0	1,2,6,7,8,0
20	1,3,5,8	1,3,4,5,6,8
18	2,3,5,0	2,3,4,5,7,0

In the first column, the intersection (extrema of a facet of S) is
named by the pair of elements missing.

To see how these faces are related, make a graph whose vertices
are the facets of S, and two vertices are adjacent iff the corresponding
facets have a three-element intersection. (See Fig. 3.1). The facet
S has nine 3-faces (tetrahedra), eighteen 2-faces (triangles), fifteen
1-faces and six extrema.

Remaining faces

The 5-element facets are simplexes, and the 6-element facets
are simplicial, so little further description is needed. The 2-faces
all have three elements. Of the $C(10,3) = 120$ three-element subsets,

all are 2-faces except thirty, which are all independent sets in P.

The 1-faces of C are all of the $C(10,2) = 45$ two-element sets, and the 0-faces are all of the 10 one-element sets.

<u>Theorem 4</u>. The graph of C is K_9.

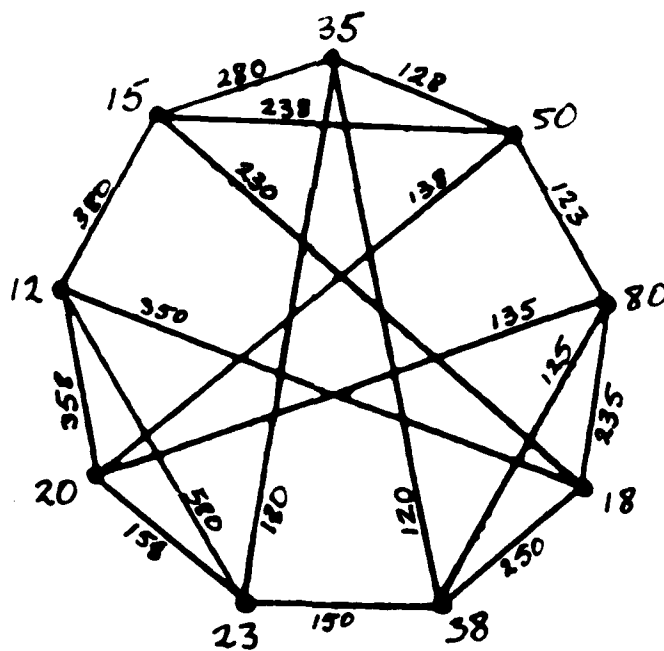It is interesting to note that the 2-element intersections of the facets of C are exactly the edges of P.

Fig. 3.1

Now multiply Eqs. (1)-(4) by e^T, and use the fact that e^T ... in ...
Eq. (5) to obtain this system (unknowns at the tops of columns):

$e^T Ce$	$e^T Cw$	$e^T De$	$e^T Dw$	$e^T Bc$	k
0	1	0	0	1	-8
1	0	0	1	0	8
1	0	0	0	1	12
1	0	1	0	0	18

Since $e^T Be$ is twice the number of edges in the graph induced by U, it must be an even integer. But $e^T Be = 0$ implies U an independent set, so we discard that case.

Case 1. $e^T Be = 2$. Then $e^T Cw = -10$, $e^T Ce = 10$, $e^T Dw = -2$, $e^T De = 8$.
Thus the six vertices in V - U induce a graph with four edges, and hence a union of two trees. Normalize by assuming 1,2 $\in$ U, 5,6 $\notin$ U.
Fig. 4.1 shows that the only way to disconnect the graph on the remaining eight vertices, without removing 5 or 6, is to remove 8 or 9 or 4 or 0. (Isomorphic.) Hence U = {1,2,8,9}.
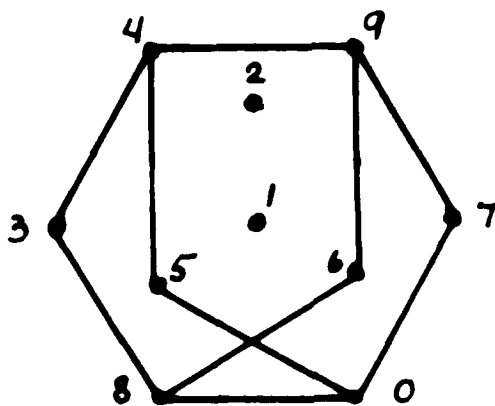

Fig 4.1

The Polytope $C' = C(-2)$

Theorem 1. If U is a set of independent vertices of P, then the corresponding rows of Z are the extrema of a facet of C.

Proof. Let $V_1 = U$ as above and $V_2 = V - U$. This partition is a coloration, with coloration matrix

$$B = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

The eigenvector of B corresponding to the eigenvalue -2 is $[3,-2]$. Thus, the rows of Z corresponding to U lie on a bounding hyperplane. #

Corollary. If W is the complement of a set U of independent vertices, then the corresponding rows of Z are also the extrema of a facet of C and lie on a parallel hyperplane. #

Theorem 2. C' has no further facets.

Proof. Let U be a set of four vertices of P corresponding to rows of Z that lie on a supporting hyperplane. Permute A so that

$$Zn = \begin{pmatrix} e \\ w \end{pmatrix} = z, \quad A = \begin{pmatrix} B & C \\ C^T & D \end{pmatrix}, \quad w \le e .$$

Then B and D are the adjacency matrices of the graphs induced by U and by its complement. We have the equations:

$$Az = -2z: \quad Be + Cw = -2e \tag{1}$$

$$C^Te + Dw = -2w \tag{2}$$

$$Ae = 3e: \quad Be + Ce = 3e \tag{3}$$

$$C^Te + De = 3e \tag{4}$$

$$e^Tz = 0: \quad e^Te + e^Tw = 0 \tag{5}$$

Case 2. $e^TBe = 4$. Then $e^TCw = -12$, $e^TCe = 8$, $e^TDe = 10$, $e^TDw = 0$.

The graph induced by U has two edges and hence is $P_3 \cup K_1$ or $2P_2$.

In the first case, we may take $1,2,5$ as forming the P_3. Then vertices

$3,4,6,7,0$ cannot be in U which must be $\cdot 1,2,5,8\cdot$ or an isomorphic

image. (Fig. 4.2.) In the second case, take $1,2 \cdot U$, $5,6 \notin U$ as

previously. But now the complement of U induces five edges, and is

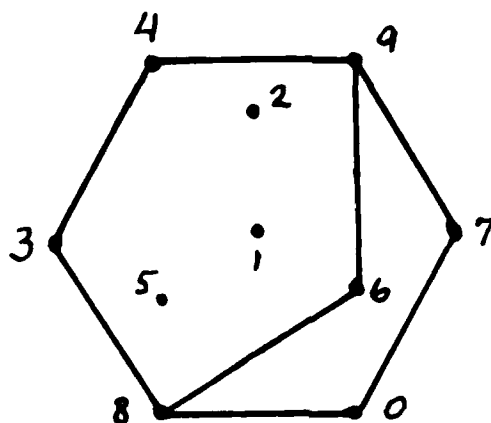
Fig 4.2

formed by deleting two adjacent vertices from the complement of $\cdot 1,2\cdot$.

The possibilities are $U = \cdot 1,2,4,9\cdot$ and $U = \cdot 1,2,8,0\cdot$ (isomorphic).

Case 3. $e^TBe = 6$. Then $e^TCw = -14$, $e^TCe = 6$, $e^TDe = 12$, $e^TDw = 2$.

The four vertices of U induce a tree: either $K_{1,3}$ or P_4. If

$\cdot U\cdot = K_{1,3}$, we may take $U = \cdot 1,2,5,6\cdot$. However, we then have $z_1 =$

$z_2 = z_5 = z_6 = 1$, so the equation $z_2 + z_5 + z_6 = -2z_1$ cannot be fulfilled.

If $\cdot U\cdot = P_4$, we may take $U = \cdot 1,2,3,5\cdot$.

In the Petersen graph, four vertices cannot induce more than

three edges, since that would require a cycle, and the girth is 5.

The four sets to check are $\cdot 1,2,8,9\cdot$, $\cdot 1,2,5,8\cdot$, $\cdot 1,2,4,9\cdot$, $\cdot 1,2,3,5\cdot$.

In all but the second case, the specified set is contained in the

complement of an independent set, and thus is accounted for by the Corollary. For $U = \{1,2,5,8\}$, we are to assume $z_i = 1$, $i \in U$. Then the first row of $Az = -2z$ is

$$z_2 + z_5 + z_6 = -2z_1$$

from which $z_6 = -4$. Then the sixth row of $Az = -2z$ is

$$z_1 + z_8 + z_9 = -2z_6$$

from which $z_9 = 6$. Thus if an eigenvector has $z_i = 1$ for $i \in U$, it does not have $z_i \leq 1$ for $i \notin U$. #

The polytope C' has ten facets: five pairs consisting of a 4-element facet and a 6-element facet. The 4-element facets are obviously tetrahedra. The 6-element facets are octahedra; this can be shown by translating one to the origin and drawing it.

Each tetrahedral facet corresponds to an independent set in P and thus cannot intersect another in more than one extremum. The facets and their 3-element intersections are shown in Table 4.2.

Table 4.2

							$O1$	$O2$	$O3$	$O4$	$O5$
$O1$	1	2	4	8	9	0	*				
$O2$	1	3	4	6	7	0	140	*			
$O3$	1	3	5	7	8	9	189	137	*		
$O4$	2	3	5	6	9	0	290	360	359	*	
$O5$	2	4	5	6	7	8	248	467	578	256	*
$T1$	3	5	6	7			-	136	357	356	567
$T2$	2	5	9	8			289	-	589	259	258
$T3$	2	4	6	0			240	460	-	260	246
$T4$	1	4	7	8			148	147	178	-	478
$T5$	1	3	9	0			190	130	139	390	-

There are a total of 30 different 2-faces. Direct computation show that these intersect pairwise on 30 1-faces, which are exactly all the possible sets of two elements except those sets that are edges of P.

Theorem 3. The graph of C' is $\bar{P}$. #

The relations among the faces of C' can be represented by means of the graph in Fig. 4.3. The graph represents C'; each vertex is an octahedral or tetrahedral facet; two vertices are adjacent iff the corresponding facets have a 3-element intersection. Thus, each edge corresponds to a 2-face of C': eight issue from each octahedral vertex and four from each tetrahedral vertex. Furthermore, each 3-cycle corresponds to a 1-face or edge of C'. For example $O1$, $F4$ and $O2$ lie on a 3-cycle. The three edges correspond to the 2-faces 148, 147 and 140. Each pair of these 2-faces intersects on the 1-face 14.
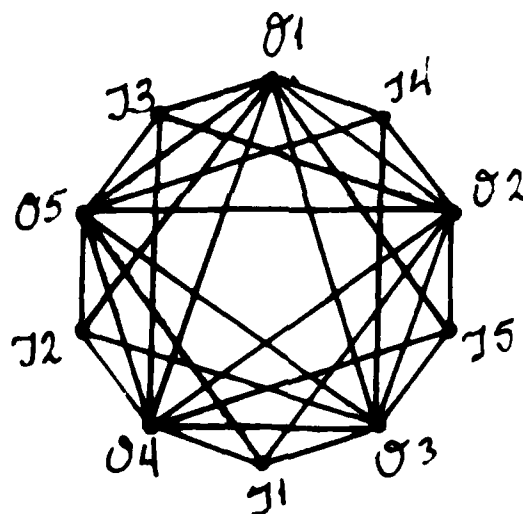

Fig 4.3

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